No. of Printed Pages : 5

5-SEMAS-Math-C-12(R&B)

2023

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*

Part of each question should be answered continuously.

Figures in the right hand margin indicate marks.

The symbols used have their usual meaning.

GROUP – A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) A set containing of a single non-zero vector is ________.

(LI or LD)

(b) Every change of coordinate matrix is invertible.
(Write true or false.)

(c) A is invertible if and only if L_A is invertible.
(Write true or false.)

(d) The eigen value of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ are ________ and ________

(e) Every linear transformation is a linear functional.
(Write true or false.)

(f) If $\langle X, Y \rangle = 0$ for all X in an inner product space, then Y =
________.

P.T.O.

(g) Every vector space is isomorphic to its dual space. (Write true or false.)

(h) Every self adjoint operator is normal. (Write true or false.)

(i) Find the value of m such that (m, 7, –4) is linear combination of vectors (1, 2, 3) and (1, 1, 1).

(j) Let V be vector space with dim n. Any linearly independent subset of V that contains _______ number of vectors is a basis for V.

(k) A matrix A is said to be self adjoint if A = ________ .

(l) The eigen values of an orthogonal matrix are always _______

GROUP – B

2. Answer <u>any eight</u> of the following questions. [2 × 8

(a) In a vector space V(F), show that

$$(-a)x = -(ax) = a(-x) \ \forall \ a \in F, x \in V.$$

(b) Show that {0} is a subspace of the vector space V over F.

(c) Define span of a set.

(d) Define anhilator.

(e) Show that self-adjoint operators are normal.

(f) Find the basis of

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in R^5 : a_2 = a_3 = a_4, a_1 + a_5 = 0\} \text{ of } R^5.$$

(g) Show that the vector (1, 1, 0), (1, 0, 1), (0, 1, 1) generates R^3.

(h) Find a matrix A, whose minimal polynomial is $t^3 - 5t^2 + 6t + 8$.

(i) If $V = R^3$ and $S = \{e_3\}$, then show that $S^{\perp}$ = XY-plane.

(j) Find the orthogonal projection of the vector u = (2, 6) on the subspace W = {(x, y) : y = 4x} of the inner product space V = R^2.

GROUP – C

3. Answer <u>any eight</u> questions [3 × 8

(a) Prove that the span of any subset S of a vector space V is a subspace of V.

(b) Let V and W be vector spaces over a field F and let T : V → W be linear. Prove that for all a ∈ F, aT + U is linear.

(c) Let β = {(2, 1), (3, 1)} be an ordered basis for R^2. Let the dual basis β is given by $\beta^* = \{f_1, f_2\}$. Find the formula for f_1 and f_2.

(d) Let $A \in M_{m \times n}(F)$. Then prove that $\text{rank}(A^* A) = \text{Rank } A$.

(e) Define $T : R^3 \to R^2$ by $T(x_1, x_2) = (x_1 + 1, x_2 + x_3)$.
Is T linear ? Justify your answer.

(f) Show that S^0 is a subspace of V^*.

(g) Apply Gram-Schimdt process to the subset $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$ of the inner product space $V = R^3$ to obtain an orthogonal basis for span (S).

(h) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -6 & -2 \end{bmatrix} \text{ and find its inverse.}$$

(i) State Spectral theorem.

(j) Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if TU = UT.

GROUP – D

*Answer **all** questions.*

4. Let $S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$. Find a basis for R^3 that is subset of S. [7

OR

Suppose that $T : R^2 \to R^2$ is linear $T(1, 0) = (1, 4)$, $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-one ?

5. Let V and W be finite-dimensional vector spaces (over the same field). Then prove that V, is isomorphic to W iff dim(V) = dim(W). [7

OR

Let $V = R^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$$f_1(x, y, z) = x - 2y,\ f_2(x, y, z) = x + y + z,\ f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for and V^* and then find a basis for V for which it is the dual space.

6. Let T be a linear operator on R^3 defined by [7

$$T(a_1, a_2, a_3) = (4a_1 + a_3,\ 2a_1 + 3a_2 + 2a_3,\ a_1 + 4a_3).$$

Test T for diagonalizability.

OR

State and prove Cayley-Hamilton theorem.

7. Let $V = P_3(R)$ and for $f, g \in V$, $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$. [7

Find the orthogonal projection of $f(x) = x^3$ on $P_2(R)$.

OR

Find the minimal solution using adjoint matrix for the following system of linear equations :

$$x + 2y + z = 4,\ x - y + 2z = -11,\ x + 5y = 19.$$