2023

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*

Part of each question should be answered continuously.

Figures in the right hand margin indicate marks.

The symbols used have their usual meaning.

GROUP – A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) If P(A) has 256 elements, then how many elements are there in A ?

(b) Write the contrapositive of $x^2 = 1 \Rightarrow x = \pm 1$.

(c) Give an example of total ordering relation.

(d) What is range of function $f(x) = x^{-1}$ which is defined everywhere on its domain.

(e) What is the value of $\phi(35)$?

(f) If gcd(a, c) = gcd(b, c) = 1, then gcd(ab, c) = ________.

(g) A number of the form $2^{2n} + 1$ for non-negative integer n is a ________.

(h) Find the number of divisors of 7056.

(i) If an inverse of a matrix A exists, then it is ________.

(j) If λ is an eigen value of the matrix A, then the eigen value of A^n is ________.

(k) A basis of a vector space is maximal L.I. subset of that vector space. (Write true or false.)

(l) Let $T : V \to U$ be a linear map. Then $T(0_v) =$ ________.

GROUP – B

2. Answer any eight of the following questions. [2 × 8

(a) Let A, B, C be subsets of some universal set U. Then prove that

$$A \cap B \subseteq C \text{ and } A^C \cap B \subseteq C \to B \subseteq C.$$

(b) Give an example of a relation on a set is systematric and anti-symmetric.

(c) Define partial order.

(d) Show that $2^{15} - 1$ is not a prime number.

(e) Solve for $x : 2x \equiv 18 (\text{mod } 50)$.

(f) If $\gcd(n, n + 1) = 1$ for any $n \in N$, then find integers x and y such that $nx + (n + 1)y = 1$.

(g) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(h) Prove that the determinant of an idempotent matrix is either 0 or 1.

(i) If V has a basis of n elements, then prove that every set of p vectors with $p > n$ is LD.

(j) If $T : U \to V$ be a linear map where U is finite dimensional, then prove that $n(T) \le \dim U$.

GROUP – C

3. Answer any eight questions [3 × 8

(a) For integers a, b, define $a \sim b$ iff $2a + 3b = 5n$ for some integer n, show that ~ defines an equivalence relation on Z.

(b) Show that the compound statement

$$[p \vee ((\sim r) \to (\sim s))] \vee [s \to ((\sim t) \vee p)) \vee ((\sim q) \to r)]$$

is neither a tautology nor a contradiction.

(c) Prove that the set of transcendental number is uncountable.

(d) Prove that $2^{5n+1} + 5^{n+2}$ is divisible by 27 for any positive integer n.

(e) Find the gcd of 243 and 198 by Euclidean algorithm and express it in the form $198m + 243n$.

(f) If A is non-singular square matrix, then prove that A^T is also non-singular and also prove that $(A^T)^{-1} = (A^{-1})^T$.

(g) Determine eigen values of the matrix

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(h) Prove that the intersection of two subspaces of V is a sub-space of V.

(i) Check whether the set of vectors $\{(1, 0, 1), (1, 1, 0), (1, -1, 1), (1, 2, -3)\}$ is LD or LI.

(j) Let $T : V_3 \to V_3$ be defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$. Prove that T is a linear map.

GROUP – D

*Answer **all** questions.*

4. Prove that the union of a family of countable sets is countable. [7

OR

If $f : A \to B$ is one-to-one and onto, then prove that the inverse mapping of f is unique.

5. State and prove Division algorithm. [7

OR

By using the Mathematical induction, prove that the truth of $7^{2n} + 2^{3n-3} . 3^{n-1}$ is divisible by 25, for all $n \geq 1$.

6. Solve the following system of linear equations by using the row-reduction method : [7

$$x - y + z = 0$$
$$2x + y - 3z = 1$$
$$-x + y + 2z = -1$$

OR

Find the range, rank, kernel and nullity of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 7 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

7. In a vector space V, if $\{v_1, v_2, \ldots, v_n\}$ generates V and if $\{w_1, w_2, \ldots, w_m\}$ is LI, then prove that $m \leq n$. [7

OR

State and prove the rank nullity theorem for a linear map $T : U \to V$.