2023

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*
Part of each question should be answered continuously.
Figures in the right hand margin indicate marks.
The symbols used have their usual meaning.

GROUP – A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) The symmetry group of the equilateral triangle is represented by ________ .

(b) The order of U(12) is ________ .

(c) Every group is a subgroup of itself. (Write true or false.)

(d) An infinite cyclic group has ________ generators.

(e) The product of two odd permutations is a ________ permutation.

(f) What are the generators of Z_6 ?

(g) If G be a finite group and $a \in G$, then $a^{|G|} =$ ________ .

(h) Every subgroup of an abelian group is ________ .

(i) Factor group of a cyclic group is ________ .

(j) Every homomorphism is a one-to-one mapping. (Write true or false.)

(k) If H is abelian, then $\phi(H)$ is ________ .

(l) If ϕ is onto, then $G/\ker\phi$ ________ $\bar{G}$.

GROUP – B

2. Answer any eight of the following questions. [2 × 8

(a) Prove that if $a^2 = a$, $a \in G$, then $a = e$.

(b) Show that U(14) is a cyclic.

(c) Define centralizer.

(d) What are the order of permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 2 & 1 & 7 & 4 \end{pmatrix}.$$

(e) What are the properties that G must be satisfied in order to be a group ?

(f) State the Fundamental theorem of cyclic group.

(g) G be a group with subgroup H and K. If $|G| = 660$, $|K| = 66$ and $K \subset H \subset G$. What are the possible values of $|H|$?

(h) Define a factor group.

(i) Show that the mapping ϕ from R^* to R defined by $\phi(x) = |x|$ is a homomorphism with $\ker\phi = \{1, -1\}$.

(j) If $|a| = n$, then prove that $|\phi(a)|$ divides n.

GROUP – C

3. Answer any eight questions [3 × 8

(a) Find the inverse of the element

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} \text{ in } GL(2, Z_5).$$

(b) In a group G, prove that $(ab)^{-1} = b^{-1}a^{-1}$.

(c) If H is a subgroup of G, then show that

$C(H) = \{x \in G : xh = hx \text{ for all } h \in H\}$ is a subgroup of G.

(d) Prove that every permutation can either be even or odd but not both.

(e) Prove that S_n is non-abelian for all $n \geq 3$.

(f) Find the order of (3, 10, 9) in $Z_4 \oplus Z_{12} \oplus Z_{15}$.

(g) For every positive integer a and every prime p, prove that $a^p \equiv a \pmod p$.

(h) Show that A_3 is normal in S_3.

(i) Show that $3Z/12Z \approx Z_4$.

(j) Consider Z_{35} where the group operation is addition, then show that

$$Z_{35} \approx Z_7 \oplus Z_5 \approx \langle 5 \rangle \oplus \langle 7 \rangle.$$

GROUP – D

*Answer **all** questions.*

4. Show that the set {1, 2, 3} under multiplication modulo 4 is not a group but the set {1, 2, 3, 4} under multiplication modulo 5 is a group. [7

OR

G be a group, H is a subgroup of a group G. Then prove that

$a \equiv b \bmod H$ is an equivalence relation.

5. Let H and K be finite subgroups of a group G. Then prove that

$$o(HK) = \frac{o(H)\,o(K)}{o(H \cap K)}.$$ [7

OR

Prove that every permutation of a finite set can be written as a product of disjoint cycles.

6. Prove that a subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H in G is again a right coset of H in G. [7

OR

Prove that U(n) is expressed as an external direct product.

7. State and prove Cauchy's theorem for a finite abelian group. [7

OR

State and prove First Isomorphism theorem.