2023

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*

Part of each question should be answered continuously.

Figures in the right hand margin indicate marks.

The symbols used have their usual meaning.

GROUP – A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) Identify the type of Indeterminate form in $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ at x = 0.

(b) $F(x) = x^2 e^{x+1}$ has a local minimum at which point ?

(c) Taylor's theorem can be regarded as an _________ of the Lagrange's Mean Value theorem.

(d) The necessary condition for the Maclaurin's expansion to be true for the function f(x) should be _________.

(e) If f is bounded and integrable on [a, b], then there exists a number λ lying between the bounds of f such that $\int_a^b f\,dx =$ _________.

(f) Every continuous function on [a, b] is integrable. (Write true or false.)

(g) Give an example of a function which is Riemann integrable but not monotonic.

(h) If $f(x) \le g(x)$ for all $x \in (a, b)$ and $\int_a^b f(x)\,dx$ is divergent, then $\int_a^b g(x)dx$ is ________ .

(i) Continuity of the limit function does not ensure the ________ of sequence of functions.

(j) If $M_n = \sup\{(a_k), k \ge n\}$, then $\lim_{n \to \infty} \sup a_n =$ ________ .

(k) A power series converges for $|x| < \rho$ and diverges for $|x| > \rho$, then f is known as ________ .

(l) If $\lim_{n \to \infty} \left| \frac{a_n + 1}{a_n} \right| = 0$, then the series is convergent for all x and $\rho =$ ________ .

GROUP – B

2. Answer any eight of the following questions. [2 × 8

(a) By using Taylor's theorem, find a polynomial f(x) of degree 2 which satisfies

$$f(1) = 2,\ f'(1) = -1 \text{ and } f''(1) = 2.$$

(b) Yse L' Hospital rule to show that $\lim_{x \to 0} \frac{x - \log(1 + x)}{1 - \cos x} = 1.$

(c) State the condition for extrema.

(d) Give an example of f and g which are not integrable such that f + g is integrable.

(e) Let $f : [0, 1] \to R$ be defined by $f(0) = 0$, $f(x) = [x^{-1}]^{-1}$, $x \ne 0$. Show that $f(x) \in R[0, 1]$.

(f) Define improper integral.

(g) If $f_n(x) = \frac{\sin x}{\sqrt{n}}$, then find f(x).

(h) Write the p-test for convergence.

(i) State the Weirstrass' M-test for series.

(j) Show that $\int_0^1 \sum_{k=0}^{\infty} \frac{x^k}{k!}\,dx = e - 1.$

GROUP – C

3. Answer any eight questions [3 × 8

(a) If f'' is continuous on $[a - \delta, a + \delta]$ for some $\delta > 0$, then show that

$$\lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a).$$

(b) Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{\tan 3x}{\tan x}$.

(c) Find the maximum and minimum of

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x,\ x \in [0, \pi].$$

(d) Show that $\left| \int_p^q \frac{\sin x}{x}\, dx \right| \le \frac{2}{p}$, $q > p > 0$.

(e) If $f \in B[a, b]$ and if f is Darboux integrable on $[a, b]$, then prove that for all $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

(f) Show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{k^2 + n^2} = \log \sqrt{2}.$$

(g) Prove that $B(m, n) = B(m, n + 1) + B(m + 1, n)$.

(h) Show that (x^n) is not uniformly convergent on $[0, 1]$.

(i) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(j) Find the radius of convergence of the series

$$1 + \frac{x}{2} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \ldots\ldots\ldots\ldots$$

GROUP – D

*Answer **all** questions.*

4. State and prove Cauchy's mean value theorem with geometrical interpretation. [7

OR

Expand $\log(1 + x)$ by Maclaurin's series.

5. Prove that every continuous function is integrable. [7

OR

Test the integrability of $f(x) = \begin{cases} 2x & , x \in \left[0, \frac{1}{4}\right] \\ 1 - x & , x \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ 1 + x & , x \in \left[\frac{1}{2}, 1\right] \end{cases}$ on $[0, 1]$.

6. Prove that the improper integral $\beta(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\, dt$ converges for $p > 0$, $q > 0$. [7

OR

If $f_n(x) = |x|^{1 + \frac{1}{n}}$, $x \in [-1, 1]$, then show that $f_n \in D[-1, 1]$, $f_n(x) \to f(x) = |x|$ uniformly on $[-1, 1]$ and f is not differentiable.

7. State and prove generalised Abel's theorem. [7

OR

Prove that a power series can be differentiated term by term strictly within its interval of convergence.